

ENDOSCOPY AND COHOMOLOGY OF A QUASI-SPLIT $U(4)$

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ABSTRACT. We prove asymptotic upper bounds for the L^2 Betti numbers of locally symmetric spaces associated to a quasi-split $U(4)$. In degree 3, we conjecture that these bounds are sharp. Our main tool is the endoscopic classification of automorphic representations of $U(N)$ by Mok.

1. INTRODUCTION

Let E be an imaginary quadratic field. Let $N \geq 1$, let $U(N)$ be the quasi-split unitary group of degree N with respect to E/\mathbb{Q} , and let G be an inner form of $U(N)$. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic congruence lattice, and for $n \geq 1$ let $\Gamma(n)$ be the corresponding principal congruence subgroup of Γ . Let K_∞ be a maximal compact subgroup of $G(\mathbb{R})$, and let $Y(n) = \Gamma(n) \backslash G(\mathbb{R}) / K_\infty$. This article is interested in how the L^2 Betti numbers $h_{(2)}^i$ of $Y(n)$ grow with n , specifically in the case when $G = U(4)$. We let $V(n) = |\Gamma : \Gamma(n)|$, which is asymptotically equal to the volume of $Y(n)$. The standard bound that we wish to improve over is $h_{(2)}^i(Y(n)) \ll V(n)$, which follows immediately from Hodge theory if Γ is cocompact, and otherwise from the results of [3] expressing $h_{(2)}^i(Y(n))$ in terms of automorphic representations, together with Savin's bound [12] for the multiplicity of a representation in the cuspidal spectrum and Langlands' theory of Eisenstein series.

The basic principle that we shall use to bound $h_{(2)}^i(Y(n))$ is the fact that, if i is not half the dimension of $Y(n)$, the archimedean automorphic forms that contribute to $h_{(2)}^i(Y(n))$ must be nontempered. In the case where Γ is cocompact, one may combine this principle with the trace formula and basic results about matrix coefficients to prove a bound of the form $h_{(2)}^i(Y(n)) \ll V(n)^{1-\delta}$ for some $\delta > 0$. In [11], Sarnak and Xue suggest the optimal bound that one should be able to prove in this way using only the archimedean trace formula. In the case when $N = 3$ and Γ is cocompact, they predict that $h_{(2)}^1(Y(n)) \ll_\epsilon V(n)^{1/2+\epsilon}$, while they prove that $h_{(2)}^1(Y(n)) \ll_\epsilon V(n)^{7/12+\epsilon}$.

There is a deeper way in which one may exploit nontemperedness to prove bounds for cohomology. In [9] Mok, following Arthur [1], classifies the automorphic spectrum of $U(N)$ in terms of conjugate self-dual cusp forms on GL_M/E for $M \leq N$. One of the implicit features of this classification is that if a representation π on $U(N)$ is sufficiently nontempered at one place, then it must be built up from cusp forms on groups GL_M/E with M strictly less than N – in other words, π comes from a smaller group. We have been interested in deriving quantitative results from this qualitative feature of the classification. In [7], we used this (more precisely, the complete solution of endoscopy for $U(3)$ by Rogawski in [10]) to prove that $h_{(2)}^1(Y(n)) \ll_\epsilon V(n)^{3/8+\epsilon}$ when $N = 3$ and G is arbitrary, strengthening the bound of

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Sarnak and Xue. Moreover, we proved that this bound is sharp. In this article, we partially extend this result to the case $G = U(4)$.

Theorem 1.1. *If $G = U(4)$ and $i = 2$ or 3 , and n is only divisible by primes that split in E , we have $h_{(2)}^i(Y(n)) \ll_{\epsilon} V(n)^{8/15+\epsilon}$.*

See Theorem 2.1 for a precise statement. As Theorem 1.1 relies on the results of Mok in [9], it is conditional on the stabilization of the twisted trace formula that is currently being carried out by Mœglin and Waldspurger [8, 14, 15, 16, 17, 18].

We expect Theorem 1.1 to be sharp in the case $i = 3$, but when $i = 2$ we expect the true order of growth to be $V(n)^{2/5+\epsilon}$. The reason for this discrepancy is that, assuming the Adams-Johnson conjectures on the structure of cohomological Arthur packets, the main contribution to $h_{(2)}^2$ comes from parameters of the form $\nu(2) \boxtimes \phi^N$ with $\phi^N \in \tilde{\Phi}_{\text{ell}}(2)$ (in the notation of §2.5). We do not have sharp bounds for the contribution from these parameters, because we do not have sharp bounds for the dimensions of spaces of K -fixed vectors in Speh representations as K shrinks. Note that we have $h_{(2)}^1(Y(n)) = 0$ for all n , by combining the noncompact Matsushima formula contained in [3] with the vanishing theorems of e.g. §10.1 of [4]. The results of [12] also imply that $h_{(2)}^4(Y(n)) \gg V(n)$.

As in [7], the main task in the proof of Theorem 1.1 is to bound the dimensions of spaces of K -fixed vectors in representations occurring in the Arthur packets, and to produce bounds that can be summed over all packets. We do this in two ways. At split places, the local packets are explicitly described singletons, and it is easy to do this directly. At nonsplit places, we express the dimension of the space of K -fixed vectors as a trace, and apply the character identities in Theorem 3.2.1 of [9]. This requires the twisted fundamental lemma, and the reason we restrict ourselves to full level at nonsplit places is that we do not know the twisted FL for Lie algebras and hence can only control the transfer of principal congruence subgroups of full level. However, it should be possible to prove the twisted FL for Lie algebras by following Waldspurger's proof for groups in [13].

The tools used in the proof should extend to a general $U(N)$ with a little extra work. However, because the recipe for the degrees of cohomology on $U(N)$ to which an Arthur parameter can contribute is complicated, the result this would give for cohomology growth would not be as strong.

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2. NOTATION

2.1. Number fields. Let E be an imaginary quadratic field with ring of integers \mathcal{O} . We denote the conjugation of E over \mathbb{Q} by c . We denote the adeles of \mathbb{Q} and E by \mathbb{A} and \mathbb{A}_E respectively. We denote places of \mathbb{Q} and E by v and w respectively. We fix a character χ of $E^\times \backslash \mathbb{A}_E^\times$ whose restriction to \mathbb{A}^\times is the character associated to E/\mathbb{Q} by class field theory. We let S_f be a finite set of finite places of \mathbb{Q} that contains all finite places at which E is ramified, and all finite places dividing a place of E at which χ is ramified. If F is a local or global field, we denote its Weil group by W_F . If F is local, we let L_F denote its local Langlands group, which is given by W_F if F is archimedean and $W_F \times SU(2)$ otherwise.

2.2. Algebraic groups. For any $N \geq 1$, we let $U(N)$ denote the quasi-split unitary group over \mathbb{Q} with respect to E/\mathbb{Q} , whose group of \mathbb{Q} -points is

$$U(N)(\mathbb{Q}) = \{g \in GL(N, E) \mid {}^t c(g)Jg = J\}$$

where

$$J = \begin{pmatrix} & & & 1 \\ & & \ddots & \\ & & & \\ 1 & & & \end{pmatrix}.$$

If p splits in E as $w\bar{w}$, then $E \times_{\mathbb{Q}} \mathbb{Q}_p \simeq E_w \times E_{\bar{w}}$ and we have

$$U(N)(\mathbb{Q}_p) = \{(g_1, g_2) \in GL(N, E_w) \times GL(N, E_{\bar{w}}) \mid g_2 = J^t c(g_1)^{-1} J^{-1}\}.$$

Projection onto the first factor defines an isomorphism $U(N)(\mathbb{Q}_p) \simeq GL(N, E_w)$.

We define $G(N) = \text{Res}_{E/\mathbb{Q}} GL(N)$. We let θ denote the automorphism of $G(N)$ whose action on \mathbb{Q} -points is given by

$$\theta(g) = \Phi_N {}^t c(g)^{-1} \Phi_N^{-1} \quad \text{for } g \in G(N)(\mathbb{Q}) \simeq GL(N, E),$$

where

$$\Phi_N = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & & & \\ (-1)^{N-1} & & \ddots & \end{pmatrix}.$$

We define $\tilde{G}^+(N) = G(N) \rtimes \langle \theta \rangle$, and let $\tilde{G}(N)$ denote the $G(N)$ -bitorsor $G(N) \rtimes \theta$.

If G is an algebraic group over \mathbb{Q} or \mathbb{Q}_v , we denote $G(\mathbb{Q}_v)$ by G_v , and let $\mathcal{H}(G_v) = C_0^\infty(G_v)$. We use analogous notation for groups over E . For any $N \geq 1$ we let $\tilde{\mathcal{H}}_p(N) = C_0^\infty(\tilde{G}(N)_p)$. We fix Haar measures on $U(N)_p$ and $\tilde{G}(N)_p$ for all $N \geq 1$ and all p , subject to the condition that these measures assign volume 1 to a hyperspecial maximal compact when the groups are unramified. All traces and twisted traces will be defined with respect to these measures.

We shall denote the infinitesimal character of an irreducible admissible representation of $U(N)_\infty$ and $GL(N, \mathbb{C})$ by a point in \mathbb{C}^N/S_N and $(\mathbb{C}^N/S_N) \times (\mathbb{C}^N/S_N)$ respectively, where S_N is the symmetric group.

2.3. L -groups and embeddings. We have ${}^L U(N) = GL(N, \mathbb{C}) \rtimes W_{\mathbb{Q}}$, where $W_{\mathbb{Q}}$ acts through its quotient $\text{Gal}(E/\mathbb{Q})$ via the automorphism

$$g \mapsto \Phi_N {}^t g^{-1} \Phi_N^{-1}.$$

We have ${}^L G(N) = (GL(N, \mathbb{C}) \times GL(N, \mathbb{C})) \rtimes W_{\mathbb{Q}}$, where $W_{\mathbb{Q}}$ acts through $\text{Gal}(E/\mathbb{Q})$ by switching the two factors.

We let $\mathcal{E}(U(N))$ denote the set of isomorphism classes of endoscopic data for $U(N)$. We let $\tilde{\mathcal{E}}(N)$ denote the set of isomorphism classes of twisted endoscopic data for $\tilde{G}(N)$. The subset $\tilde{\mathcal{E}}_{\text{sim}}(N)$ of simple data contains two elements that we shall denote $(U(N), \xi_+)$ and $(U(N), \xi_-)$. Here, ξ_+ is the (untwisted) base change embedding with respect to the trivial character of $E^\times \backslash \mathbb{A}_E^\times$, and ξ_- is the (twisted) embedding with respect to χ , as defined in §2.1 of [9] or §4.7 of [10].

2.4. Local parameters. If G is a connected reductive algebraic group over a local field F , we let $\Phi(G)$ denote the set of local Langlands parameters, that is admissible maps

$$\phi : L_F \rightarrow {}^L G.$$

We let $\Psi(G)$ denote the set of local Arthur parameters, that is admissible maps

$$\psi : L_F \times SL(2, \mathbb{C}) \rightarrow {}^L G$$

such that the image of L_F in \widehat{G} is bounded. We let $\Psi^+(G)$ denote the set of parameters obtained by dropping this boundedness condition. If $\psi \in \Psi^+(U(N) \times_{\mathbb{Q}} \mathbb{Q}_v)$, we let Π_{ψ} denote the packet of representations of $U(N)_v$ defined in Theorem 2.5.1 of [9] and the subsequent remarks.

We denote $\Phi(G(N) \times_{\mathbb{Q}} \mathbb{Q}_v)$ and $\Psi(G(N) \times_{\mathbb{Q}} \mathbb{Q}_v)$ by $\Phi_v(N)$ and $\Psi_v(N)$ respectively. In §3.2 of [9], Mok associates to any $\psi^N \in \Psi_v(N)$ an irreducible unitary representation of $G(N)_v$, denoted π_{ψ^N} . He also defines a canonical extension of π_{ψ^N} to $\widetilde{G}(N)_v^+$, denoted $\widetilde{\pi}_{\psi^N}$.

2.5. Global parameters. For $n \geq 1$, we let $\nu(n)$ denote the unique irreducible algebraic representation of $SL(2, \mathbb{C})$ of dimension n . We let $\Psi_{\text{sim}}(N)$ denote the set of simple global Arthur parameters, that is, formal expressions $\psi^N = \mu \boxtimes \nu$ where μ is a unitary cuspidal automorphic representation of $GL(m, \mathbb{A}_E)$ and ν is the irreducible representation of $SL(2, \mathbb{C})$ of dimension n , and $N = mn$. We let $\Psi(N)$ denote the set of all global parameters, that is formal expressions

$$\psi^N = l_1 \psi_1^{N_1} \boxplus \cdots \boxplus l_r \psi_r^{N_r}$$

with $\psi_i \in \Psi_{\text{sim}}(N_i)$ and $l_1 N_1 + \cdots + l_r N_r = N$. We denote the set of conjugate self-dual parameters by $\widetilde{\Psi}(N)$, and have the usual chain of subsets $\widetilde{\Psi}_{\text{sim}}(N) \subseteq \widetilde{\Psi}_{\text{ell}}(N) \subseteq \widetilde{\Psi}(N)$. We denote the set of generic parameters, that is those for which all the representations ν are trivial, by $\Phi(N)$. If $\phi^N \in \Phi(N)$ and w is a place of E , the local Langlands correspondence allows us to associate maps $\phi_w^N : L_{E_w} \rightarrow GL(N, \mathbb{C})$ to ϕ^N .

To any parameter $\psi^N \in \widetilde{\Psi}(N)$, Mok associates a group \mathcal{L}_{ψ^N} which is an extension of $W_{\mathbb{Q}}$ by a complex algebraic group, and an L -homomorphism $\widetilde{\psi}^N : \mathcal{L}_{\psi^N} \times SL(2, \mathbb{C}) \rightarrow {}^L G(N)$. If $(U(N), \xi_{\pm}) \in \widetilde{\mathcal{E}}_{\text{sim}}(N)$, we define $\Psi(U(N), \xi_{\pm})$ to be the set of pairs $\psi = (\psi^N, \widetilde{\psi})$, where $\psi^N \in \widetilde{\Psi}(N)$ and

$$\widetilde{\psi} : \mathcal{L}_{\psi^N} \times SL(2, \mathbb{C}) \rightarrow {}^L U(N)$$

is an L -homomorphism such that $\widetilde{\psi}^N = \xi_{\pm} \circ \widetilde{\psi}$. If $\psi = (\psi^N, \widetilde{\psi}) \in \Psi(U(N), \xi_{\pm})$, we set $\mathcal{L}_{\psi} = \mathcal{L}_{\psi^N}$ and $\pi_{\psi_v} = \pi_{\psi_v^N}$ for any v . If $\psi \in \Psi(U(N), \xi_{\pm})$, we shall let $\psi_v \in \Psi^+(U(N) \times_{\mathbb{Q}} \mathbb{Q}_v)$ be the local parameters associated to ψ in §2.3 of [9]. We define the global packet $\Pi_{\psi} = \otimes_v \Pi_{\psi_v}$. We define $\Psi_2(U(N), \xi_{\pm})$ to be the subset of $\Psi(U(N), \xi_{\pm})$ for which $\psi^N \in \widetilde{\Psi}_{\text{ell}}(N)$.

If $\phi^N \in \widetilde{\Phi}_{\text{sim}}(N)$ is associated to a conjugate self-dual cusp form μ , Theorem 2.4.2 of [9] states that there is a unique base change map ξ_{κ} with $\kappa = \pm$ such that μ is the weak base change of a representation of $U(N)$ under ξ_{κ} . Following Mok, we refer to $\kappa(-1)^{N-1}$ as the parity of ϕ^N and μ .

2.6. Adelic groups. We choose a compact open subgroup $K = \otimes_p K_p \subset U(4)(\mathbb{A}_f)$, subject to the conditions that $K_p \subseteq U(4)(\mathbb{Z}_p)$ for all p , and $K_p = U(4)(\mathbb{Z}_p)$ for $p \notin S_f$. For any $n \geq 1$ that is relatively prime to S_f , we define $K_p(n)$ to be the subgroup of K_p consisting of elements congruent to 1 modulo n , and define $K(n) = \otimes_p K_p(n)$.

We let K_∞ be the standard maximal compact subgroup of $U(4)_\infty$. For any $n \geq 1$ that is relatively prime to S_f , we define $Y(n) = U(4)(\mathbb{Q}) \backslash U(4)(\mathbb{A}) / K_\infty K(n)$. For any $0 \leq i \leq 8$, we let $h_{(2)}^i(Y(n))$ denote the dimension of the space of square integrable harmonic i -forms on $Y(n)$. The precise statement we shall prove is the following.

Theorem 2.1. *If $i = 2, 3$, and n is relatively prime to S_f and divisible only by primes that split in E , we have $h_{(2)}^i(Y(n)) \ll n^9$.*

The implied constant depends only on K , and we shall ignore the dependence of implied constants on K for the rest of the paper. By considering the action of the center on the connected components of $Y(n)$, Theorem 2.1 implies that the connected component $Y^0(n)$ of the identity satisfies $h_{(2)}^i(Y^0(n)) \ll_\epsilon n^{8+\epsilon}$. This implies Theorem 1.1 when combined with the asymptotic $\text{Vol}(Y^0(n)) = n^{15+o(1)}$.

3. APPLICATION OF THE GLOBAL CLASSIFICATION

We shall only prove Theorem 2.1 in the case $i = 3$, as the case $i = 2$ is identical. We begin by applying the extension of Matsushima's formula to noncompact quotients that is contained in [3], which gives

$$(1) \quad h_{(2)}^3(Y(n)) = \sum_{\pi \in L_{\text{disc}}^2(U(4)(\mathbb{Q}) \backslash U(4)(\mathbb{A}))} h^3(\mathfrak{g}, K; \pi_\infty) \dim \pi_f^{K(n)}.$$

We now apply the classification of representations of $U(4)$ given by Theorem 2.5.2 of [9]. This states that

$$L_{\text{disc}}^2(U(4)(\mathbb{Q}) \backslash U(4)(\mathbb{A})) = \bigoplus_{\psi \in \Psi_2(U(4), \xi_+)} \bigoplus_{\pi \in \Pi_\psi(\epsilon_\psi)} \pi,$$

where $\Pi_\psi(\epsilon_\psi) \subseteq \Pi_\psi$; the definition of this subset will not matter to us. Applying the classification to (1) gives

$$h_{(2)}^3(Y(n)) \leq \sum_{\psi \in \Psi_2(U(4), \xi_+)} \sum_{\pi \in \Pi_\psi} h^3(\mathfrak{g}, K; \pi_\infty) \dim \pi_f^{K(n)}.$$

It follows from Proposition 13.4 of [2] that if $\pi \in \Pi_\psi$ satisfies $h^3(\mathfrak{g}, K; \pi_\infty) \neq 0$, then ψ is not generic. It follows that ψ^N must be of one of the following types.

- (a) $\nu(2) \boxtimes \phi_1^N \boxplus \phi_2^N$, $\phi_i^N \in \tilde{\Phi}_{\text{ell}}(i)$.
- (b) $\nu(2) \boxtimes \phi^N$, $\phi^N \in \tilde{\Phi}_{\text{ell}}(2)$.
- (c) $\nu(3) \boxtimes \phi_1^N \boxplus \phi_2^N$, $\phi_i^N \in \tilde{\Phi}(1)$.
- (d) $\nu(4) \boxtimes \phi^N$, $\phi^N \in \tilde{\Phi}(1)$.

We bound the contribution of parameters of types (a) and (b) in §4 and §5 respectively. It follows from the description of the packets Π_ψ at split places that all representations contained in packets of type (d) must be characters, and these make a contribution of

$\ll_\epsilon n^{1+\epsilon}$ to $h_{(2)}^3(Y(n))$. We shall also omit the case of parameters of type (c); it may be proven that they make a contribution of $\ll_\epsilon n^{5+\epsilon}$ using the same methods as in §5.

4. THE CASE $\psi^N = \nu(2) \boxtimes \phi_1^N \boxplus \phi_2^N$

In this section, we shall prove the following.

Proposition 4.1. *We have the bound*

$$(2) \quad \sum_{\substack{\psi \in \Psi_2(U(4), \xi_+) \\ \psi^N = \nu(2) \boxtimes \phi_1^N \boxplus \phi_2^N}} \sum_{\pi \in \Pi_\psi} h^3(\mathfrak{g}, K; \pi_\infty) \dim \pi_f^{K(n)} \ll n^9.$$

We first bound the contribution from parameters for which $\phi_2^N \in \tilde{\Phi}_{\text{sim}}(2)$. We describe how to treat composite ϕ_2^N in §4.3. We note that $\psi \in \Psi_2(U(4), \xi_+)$ implies that ϕ_1^N and ϕ_2^N must be even and odd respectively.

For $i = 1, 2$, we let $K_i = \otimes_p K_{i,p}$ be a compact open subgroup of $U(i)(\mathbb{A}_f)$ such that $K_{i,p} = U(i)(\mathbb{Z}_p)$ for all $p \notin S_f$, and let $\tilde{K}_i = \otimes_w \tilde{K}_{i,w}$ be a compact open subgroup of $GL(i, \mathbb{A}_{E,f})$ such that $\tilde{K}_{i,w} = GL(i, \mathcal{O}_w)$ for all $w|p \notin S_f$. We define $\tilde{K} \subset GL(4, \mathbb{A}_{E,f})$ in a similar way. The groups $K_{2,p}$ and $\tilde{K}_{1,w}$ for $w|p \in S_f$ will be specified in the proof of Proposition 4.2, and the groups $K_{1,p}$, $\tilde{K}_{2,w}$, and \tilde{K}_w for $w|p \in S_f$ may be chosen arbitrarily. We define congruence subgroups $K_*(n)$ of these groups for n relatively prime to S_f in the usual way, and recall that n will only be divisible by primes that split in E .

We let \tilde{P} be the standard parabolic subgroup of $GL(4, E)$ with Levi $\tilde{L} = GL(2, E) \times GL(2, E)$, and let P be the corresponding standard parabolic subgroup of $U(4)$.

4.1. Controlling a single parameter. We first bound the contribution from a single Arthur parameter to (2). We therefore fix $\phi_i^N \in \tilde{\Phi}_{\text{sim}}(i)$ for $i = 1, 2$ with ϕ_1^N even and ϕ_2^N odd, and let $\psi \in \Psi(U(4), \xi_+)$ be the unique parameter with $\psi^N = \nu(2) \boxtimes \phi_1^N \boxplus \phi_2^N$. We let ϕ_i^N correspond to a conjugate self-dual cuspidal automorphic representation μ_i of $GL(i, \mathbb{A}_E)$. We assume that μ_i are tempered at all places. This assumption is not necessary, but simplifies the proof of Proposition 4.2 and will be proven to hold for all parameters that contribute to cohomology.

We have $\mathcal{L}_\psi = {}^L(U(1) \times U(2))$, and we choose the embedding $\tilde{\psi} : \mathcal{L}_\psi \times SL(2, \mathbb{C}) \rightarrow {}^L U(4)$ to be

$$\tilde{\psi} : (h_1, h_2) \times \sigma \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} h_1 a & & h_1 b \\ & h_2 & \\ h_1 c & & h_1 d \end{pmatrix} \times \sigma, \quad h_i \in GL(i, \mathbb{C}), \sigma \in W_{\mathbb{Q}}.$$

We define $s_\psi = \tilde{\psi}(e, -I)$, where $e \in \mathcal{L}_\psi$ is the identity. We define $\psi_1^N = \nu(2) \boxtimes \phi_1^N$ and $\psi_2^N = \phi_2^N$. We have $\mathcal{L}_{\psi_1^N} = {}^L U(1)$ and $\mathcal{L}_{\psi_2^N} = {}^L U(2)$, and we define

$$\begin{aligned} \tilde{\psi}_1 : \mathcal{L}_{\psi_1^N} \times SL(2, \mathbb{C}) &\rightarrow {}^L U(2) \\ \tilde{\psi}_1 : z \times \sigma \times A &\mapsto zA \times \sigma, \quad z \in GL(1, \mathbb{C}), \sigma \in W_{\mathbb{Q}} \end{aligned}$$

and

$$\begin{aligned}\tilde{\psi}_2 &: \mathcal{L}_{\psi_2^N} \times SL(2, \mathbb{C}) \rightarrow {}^L U(2) \\ \tilde{\psi}_2 &: z \times \sigma \times A \mapsto z \times \sigma, \quad z \in GL(2, \mathbb{C}), \sigma \in W_{\mathbb{Q}}.\end{aligned}$$

We define $\psi_i = (\psi_i^N, \tilde{\psi}_i) \in \Psi(U(2), \xi_+)$ for $i = 1, 2$. We shall prove the following bound for the contribution of Π_{ψ} to (2).

Proposition 4.2. *There is a choice of $\tilde{K}_{1,w}$ for $w|p$, $p \in S_f$, and $K_{2,p}$ for $p \in S_f$, depending only on K , such that*

$$\sum_{\pi \in \Pi_{\psi}} \dim \pi_f^{K(n)} \ll [K : (K \cap P(\mathbb{A}_f))K(n)] \dim \mu_1^{\tilde{K}_{1,w}} \sum_{\pi' \in \Pi_{\psi_2}} \dim \pi_f'^{K_{2,p}(n)}.$$

The proposition will follow from the factorisation

$$\sum_{\pi \in \Pi_{\psi}} \dim \pi_f^{K(n)} = \prod_p \sum_{\pi_p \in \Pi_{\psi_p}} \dim \pi_p^{K_p(n)}$$

and the series of lemmas below.

Lemma 4.3. *Let $p \notin S_f$ be nonsplit in E , and let $w|p$. We have*

$$\sum_{\pi_p \in \Pi_{\psi_p}} \dim \pi_p^{K_p} = \dim \mu_{1,w}^{\tilde{K}_{1,w}} \sum_{\pi'_p \in \Pi_{\psi_{2,p}}} \dim \pi_p'^{K_{2,p}}.$$

Proof. We have

$$\sum_{\pi_p \in \Pi_{\psi_p}} \dim \pi_p^{K_p} = \sum_{\pi_p \in \Pi_{\psi_p}} \text{tr}(\pi_p(1_{K_p})),$$

and we may manipulate the RHS using the local character identities in Theorem 3.2.1 of [9]. Let $(G', \xi') \in \mathcal{E}(U(4))$ be the endoscopic datum with $G' = U(2) \times U(2)$, and ξ' the L -embedding that maps $GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$ to the subgroup

$$\begin{pmatrix} * & & * \\ & * & * \\ & * & * \\ * & & * \end{pmatrix}$$

of $GL(4, \mathbb{C})$. We let $\bar{\psi}_{i,p} : L_{\mathbb{Q}_p} \times SL(2, \mathbb{C}) \rightarrow GL(2, \mathbb{C})$ be the dual group components of $\psi_{i,p}$, and let $\psi'_p \in \Psi(G' \times_{\mathbb{Q}} \mathbb{Q}_p)$ be the unique parameter whose projection to \widehat{G}' is $\bar{\psi}_{1,p} \times \bar{\psi}_{2,p}$. We then have $\psi_p = \xi' \circ \psi'_p$. It may be checked that the recipe of §3.2 of [9] associates to (ψ_p, s_{ψ}) the pair (G'_p, ψ'_p) . We recall the stable distribution $f \mapsto f^{G'}(\psi'_p)$ on $\mathcal{H}(G'_p)$ associated to ψ'_p in Theorem 3.2.1 of [9]. Applying the character identity of Theorem 3.2.1 (b) of [9] with $x = s_{\psi}$, and the fundamental lemma for the group $G' \in \mathcal{E}(U(4))$, gives

$$\sum_{\pi_p \in \Pi_{\psi_p}} \text{tr}(\pi_p(1_{K_p})) = (1_{K_{2,p}} \times 1_{K_{2,p}})^{G'}(\psi'_p).$$

Because $\psi'_p = \psi_{1,p} \times \psi_{2,p}$, the factorisation property of Theorem 3.2.1 (a) allows us to write this as

$$\sum_{\pi_p \in \Pi_{\psi_p}} \text{tr}(\pi_p(1_{K_p})) = 1_{K_{2,p}}^{U(2)}(\psi_{1,p}) 1_{K_{2,p}}^{U(2)}(\psi_{2,p}),$$

where $f \mapsto f^{U(2)}(\psi_{i,p})$ are the stable distributions on $\mathcal{H}(U(2)_p)$ associated to $\psi_{i,p}$. The packet $\psi_{2,p}$ is stable, and so we may apply Theorem 3.2.1 (b) with $x = e$ to obtain

$$1_{K_{2,p}}^{U(2)}(\psi_{2,p}) = \sum_{\pi'_p \in \Pi_{\psi_{2,p}}} \text{tr}(\pi'_p(1_{K_{2,p}})) = \sum_{\pi'_p \in \Pi_{\psi_{2,p}}} \dim \pi'^{K_{2,p}}_p.$$

We evaluate $1_{K_{2,p}}^{U(2)}(\psi_{1,p})$ by applying its definition in equation (3.2.8) of [9] with the embedding ξ chosen to be $\xi_+ : {}^L U(2) \rightarrow {}^L G(2)$. If we restrict the map

$$\xi_+ \circ \psi_{1,p} : L_{\mathbb{Q}_p} \times SL(2, \mathbb{C}) \rightarrow {}^L G(2)$$

to $L_{E_w} \times SL(2, \mathbb{C})$, it is equivalent to

$$\begin{aligned} \xi_+ \circ \psi_{1,p} : L_{E_w} \times SL(2, \mathbb{C}) &\rightarrow GL(2, \mathbb{C}) \\ \sigma \times A &\mapsto \phi_{1,w}^N(\sigma)A. \end{aligned}$$

It follows that the representation of $G(2)_p \simeq GL(2, E_w)$ associated to $\xi_+ \circ \psi_{1,p}$ is equal to $\mu_{1,w} \circ \det$. We denote the canonical extension of this representation to $\tilde{G}^+(2)_p$ by $\tilde{\pi}_1$. If we identify $\tilde{K}_{2,w}$ with a subgroup of $G(2)_p$, the twisted fundamental lemma implies that we may apply (3.2.8) of [9] with $\tilde{f} = 1_{\tilde{K}_{2,w} \rtimes \theta} \in \tilde{\mathcal{H}}_p(2)$ to obtain

$$1_{K_{2,p}}^{U(2)}(\psi_{1,p}) = \text{tr}(\tilde{\pi}_1(1_{\tilde{K}_{2,w} \rtimes \theta})).$$

Because $\theta^2 = 1$, we have

$$\text{tr}(\tilde{\pi}_1(1_{\tilde{K}_{2,w} \rtimes \theta})) = \pm \dim \tilde{\pi}_1^{\tilde{K}_{2,w}} = \pm \dim \mu_{1,w}^{\tilde{K}_{1,w}}.$$

Positivity implies that we must take the plus sign, which completes the proof. \square

Lemma 4.4. *Let $p \notin S_f$ be split in E , and let $w|p$. Let $\Pi_{\psi_p} = \{\pi_p\}$, and $\Pi_{\psi_{2,p}} = \{\pi'_p\}$. We have*

$$\dim \pi_p^{K_p(n)} = [K_p : (K_p \cap P_p)K_p(n)] \dim \mu_{1,w}^{\tilde{K}_{1,w}(n)} \dim \pi'^{K_{2,p}(n)}_p.$$

Proof. Under the identification $U(4)_p \simeq GL(4, E_w)$, π_p is isomorphic to the representation induced from the representation $\mu_{1,w} \circ \det \otimes \mu_{2,w}$ of \tilde{P}_w . The restriction of π_p to K_p is isomorphic to the induction of $\mu_{1,w} \circ \det \otimes \mu_{2,w}$ from $\tilde{P}_w \cap \tilde{K}_w$ to \tilde{K}_w . Because $\tilde{K}_w(n) \cap \tilde{L}_w = \tilde{K}_{2,w}(n) \times \tilde{K}_{2,w}(n)$, and $\dim(\mu_{1,w} \circ \det)^{\tilde{K}_{2,w}(n)} = \dim \mu_{1,w}^{\tilde{K}_{1,w}(n)}$, we have

$$\dim \pi_p^{K_p(n)} = [\tilde{K}_w : (\tilde{K}_w \cap \tilde{P}_w)\tilde{K}_w(n)] \dim \mu_{1,w}^{\tilde{K}_{1,w}(n)} \dim \pi'^{K_{2,p}(n)}_p$$

which is equivalent to the lemma.

□

Lemma 4.5. *Let $p \in S_f$, and let $w|p$. There is a choice of $\tilde{K}_{1,w}$ and $K_{2,p}$, depending only on K_p , such that*

$$\sum_{\pi_p \in \Pi_{\psi_p}} \dim \pi_p^{K_p} \ll \dim \mu_{1,w}^{\tilde{K}_{1,w}} \sum_{\pi'_p \in \Pi_{\psi_{2,p}}} \dim \pi'^{K_{2,p}}_p.$$

Proof. If p is split, this follows easily from the explicit description of Π_{ψ_p} . Assume that p is nonsplit, and continue to use the notation of Lemma 4.3. Let $\tilde{1}_{K_p} \in \mathcal{H}(G'_p)$ be a transfer of 1_{K_p} to G'_p . Reasoning as in the proof of Lemma 4.3 gives

$$\sum_{\pi_p \in \Pi_{\psi_p}} \dim \pi_p^{K_p} = dg_p(K_p)^{-1} \tilde{1}_{K_p}^{G'}(\psi'_p).$$

We may write $\tilde{1}_{K_p} = \sum f_{i,1} \times f_{i,2}$ for $f_{i,j} \in \mathcal{H}(U(2)_p)$, and the factorisation property of $f^{G'}(\psi'_p)$ gives

$$\tilde{1}_{K_p}^{G'}(\psi'_p) = \sum f_{i,1}^{U(2)}(\psi_{1,p}) f_{i,2}^{U(2)}(\psi_{2,p}).$$

We have

$$\begin{aligned} f_{i,2}^{U(2)}(\psi_{2,p}) &= \sum_{\pi'_p \in \Pi_{\psi_{2,p}}} \text{tr}(\pi'_p(f_{i,2})) \\ &\leq C(f_{i,2}) \sum_{\pi'_p \in \Pi_{\psi_{2,p}}} \dim \pi'^{K_{2,p}}_p \end{aligned}$$

if $K_{2,p}$ is chosen sufficiently small depending on $f_{i,2}$. Likewise, applying the definition of $f_{i,1}^{U(2)}(\psi_{1,p})$ shows that $f_{i,1}^{U(2)}(\psi_{1,p}) \leq C(f_{i,1}) \dim \mu_{1,w}^{\tilde{K}_{1,w}}$ if $\tilde{K}_{1,w}$ is chosen sufficiently small depending on $f_{i,1}$. As the collection of functions $f_{i,j}$ depended only on K_p , so do $\tilde{K}_{1,w}$ and $K_{2,p}$, and the constant factors. □

4.2. Summing over parameters. We now use Proposition 4.2 to control the sum of (2) over all ψ .

Lemma 4.6. *Let $\psi \in \Psi(U(4), \xi_+)$, and suppose that $\psi^N = \nu(2) \boxtimes \phi_1^N \boxplus \phi_2^N$ with $\phi_i^N \in \tilde{\Phi}_{\text{sim}}(i)$. If $\pi \in \Pi_{\psi_\infty}$ satisfies $H^*(\mathfrak{g}, K; \pi) \neq 0$, then we have*

$$\begin{aligned} \phi_{1,\infty}^N : z &\mapsto (z/\bar{z})^{\alpha'} \\ \phi_{2,\infty}^N : z &\mapsto \begin{pmatrix} (z/\bar{z})^{\alpha_1} \\ (z/\bar{z})^{\alpha_2} \end{pmatrix} \end{aligned}$$

with $\alpha' \in \{1, 0, -1\}$, $\alpha_i \in \{3/2, 1/2, -1/2, -3/2\}$, and $\alpha_1 \neq \alpha_2$.

Proof. We write

$$\begin{aligned}\phi_{1,\infty}^N : z &\mapsto z^{\alpha'} \bar{z}^{\beta'} \\ \phi_{2,\infty}^N : z &\mapsto \begin{pmatrix} z^{\alpha_1} \bar{z}^{\beta_1} & \\ & z^{\alpha_2} \bar{z}^{\beta_2} \end{pmatrix}\end{aligned}$$

with $\alpha' - \beta', \alpha_i - \beta_i \in \mathbb{Z}$. We know that the representation π_{ψ_∞} of $GL(4, \mathbb{C})$ has infinitesimal character $(\alpha' + 1/2, \alpha' - 1/2, \alpha_1, \alpha_2; \beta' + 1/2, \beta' - 1/2, \beta_1, \beta_2)$, and so any $\pi \in \Pi_{\psi_\infty}$ must have character $(\alpha' + 1/2, \alpha' - 1/2, \alpha_1, \alpha_2)$. If π is to have cohomology we must have $\{\alpha' + 1/2, \alpha' - 1/2, \alpha_1, \alpha_2\} = \{3/2, 1/2, -1/2, -3/2\}$, so that $\alpha' \in \{1, 0, -1\}$ and $\alpha_i \in \{3/2, 1/2, -1/2, -3/2\}$ with $\alpha_1 \neq \alpha_2$. Because μ_1 is a character we have $\alpha' = -\beta'$, and because μ_2 is a cusp form on $GL(2, E)$ we have $|\alpha_i + \beta_i| < 1/2$ so that $\alpha_i = -\beta_i$. This completes the proof. \square

For $i = 1, 2$, we define $\Phi_{\text{rel}}(i) \subset \tilde{\Phi}_{\text{sim}}(i)$ to be the set of parameters ϕ^N such that ϕ_∞^N satisfies the relevant constraints of Lemma 4.6. If $\phi^N \in \Phi_{\text{rel}}(2)$ is associated to a cuspidal representation μ , it follows that μ is regular algebraic, conjugate self-dual, and cuspidal, and hence tempered at all places by Theorem 1.2 of [5].

Lemma 4.6 implies that

$$(3) \quad \sum_{\substack{\psi \in \Psi_2(U(4), \xi_+) \\ \psi^N = \nu(2) \boxtimes \phi_1^N \boxplus \phi_2^N}} \sum_{\pi \in \Pi_\psi} h^3(\mathfrak{g}, K; \pi_\infty) \dim \pi_f^{K(n)} \ll \sum_{\substack{\psi^N = \nu(2) \boxtimes \phi_1^N \boxplus \phi_2^N \\ \phi_i^N \in \Phi_{\text{rel}}(i)}} \sum_{\pi \in \Pi_\psi} \dim \pi_f^{K(n)}.$$

We may apply Proposition 4.2 to the sum over Π_ψ on the RHS to obtain

$$(4) \quad \sum_{\substack{\psi^N = \nu(2) \boxtimes \phi_1^N \boxplus \phi_2^N \\ \phi_i^N \in \Phi_{\text{rel}}(i)}} \sum_{\pi \in \Pi_\psi} \dim \pi_f^{K(n)} \ll [K : (K \cap P(\mathbb{A}_f))K(n)] \\ \sum_{\phi_1^N \in \Phi_{\text{rel}}(1)} \dim \mu_1^{\tilde{K}_1(n)} \sum_{\phi_2^N \in \Phi_{\text{rel}}(2)} \sum_{\pi' \in \Pi_{\psi_2}} \dim \pi_f'^{K_2(n)},$$

where μ_1 is the automorphic character associated to ϕ_1^N .

Lemma 4.6 implies that there are only three possibilities for $\mu_{1,\infty}$, and therefore

$$(5) \quad \sum_{\phi_1^N \in \Phi_{\text{rel}}(1)} \dim \mu_1^{\tilde{K}_1(n)} \ll [K_1 : K_1(n)].$$

There is a finite set Ξ_∞ of representations of $U(2)_\infty$ such that if $\phi_2^N \in \Phi_{\text{rel}}(2)$ and $\pi' \in \Pi_{\psi_2}$, then $\pi'_\infty \in \Xi_\infty$. Moreover, because ψ_2 is a simple generic parameter, every $\pi' \in \Pi_{\psi_2}$ occurs in $L_{\text{disc}}^2(U(2)(\mathbb{Q}) \backslash U(2)(\mathbb{A}))$ with multiplicity one. If we define $X(n) = U(2)(\mathbb{Q}) \backslash U(2)(\mathbb{A}) / K_2(n)$, we therefore have

$$\begin{aligned}
\sum_{\phi_2^N \in \Phi_{\text{rel}}(2)} \sum_{\pi' \in \Pi_{\psi_2}} \dim \pi_f'^{K_2(n)} &\leq \sum_{\substack{\pi' \in L_{\text{disc}}^2(U(2)(\mathbb{Q}) \backslash U(2)(\mathbb{A})) \\ \pi'_\infty \in \Xi_\infty}} \dim \pi_f'^{K_2(n)} \\
&= \sum_{\pi_\infty \in \Xi_\infty} m(\pi_\infty, L_{\text{disc}}^2(X(n))) \\
(6) \quad &\ll [K_2 : K_2(n)].
\end{aligned}$$

Combining (3)–(6) gives

$$\sum_{\substack{\psi \in \Psi_2(U(4), \xi_+) \\ \psi^N = \nu(2) \boxtimes \phi_1^N \boxplus \phi_2^N}} \sum_{\pi \in \Pi_\psi} h^3(\mathfrak{g}, K; \pi_\infty) \dim \pi_f^{K(n)} \ll [K : (K \cap P(\mathbb{A}_f))K(n)][K_2 : K_2(n)][K_1 : K_1(n)].$$

Applying the formula for the order of $GL(N)$ over a finite field completes the proof.

4.3. The case of ϕ_2^N composite. We now briefly explain how to bound the contribution to (2) from parameters with $\phi_2^N = \phi_{21}^N \boxplus \phi_{22}^N$, where $\phi_{2i}^N \in \tilde{\Phi}(1)$. We let ϕ_{2i}^N correspond to a conjugate self-dual character μ_{2i} on $GL(1, \mathbb{A}_E)$. Let P_2 be the standard Borel subgroup of $U(2)$. We may prove the following analogue of Proposition 4.2.

Proposition 4.7. *There is a choice of $\tilde{K}_{1,w}$ for $w|p$, $p \in S_f$, depending only on K , such that*

$$\begin{aligned}
(7) \quad \sum_{\pi \in \Pi_\psi} \dim \pi_f^{K(n)} &\ll [K : (K \cap P(\mathbb{A}_f))K(n)][K_2 : (K_2 \cap P_2(\mathbb{A}_f))K_2(n)] \\
&\quad \dim \mu_1^{\tilde{K}_1(n)} \dim \mu_{21}^{\tilde{K}_1(n)} \dim \mu_{22}^{\tilde{K}_1(n)}.
\end{aligned}$$

The proof follows the same lines, by using the explicit description of π_{ψ_p} when p is split and the character identities of [9] Theorem 3.2.1 when p is inert. There are $\ll n^3$ choices for the three characters, and the coset factors in (7) make a contribution of $\ll_\epsilon n^{5+\epsilon}$. Therefore the contribution to cohomology of parameters of this type is bounded by $\ll_\epsilon n^{8+\epsilon}$ as required.

5. THE CASE $\psi^N = \nu(2) \boxtimes \phi^N$

In this section, we shall prove the following.

Proposition 5.1.

$$(8) \quad \sum_{\substack{\psi \in \Psi_2(U(4), \xi_+) \\ \psi^N = \nu(2) \boxtimes \phi^N}} \sum_{\pi \in \Pi_\psi} h^3(\mathfrak{g}, K; \pi_\infty) \dim \pi_f^{K(n)} \ll n^9.$$

We first bound the contribution from those parameters with $\phi^N \in \tilde{\Phi}_{\text{sim}}(2)$, and describe how to treat composite ϕ^N in §5.3. We note that $\psi \in \Psi_2(U(4), \xi_+)$ implies that ϕ^N must be even.

We define compact open subgroups $K' = \otimes_p K'_p \subset U(2)(\mathbb{A}_f)$, $\tilde{K}' = \otimes_w \tilde{K}'_w \subset GL(2, \mathbb{A}_{E,f})$, and $\tilde{K} = \otimes_w \tilde{K}_w \subset GL(4, \mathbb{A}_{E,f})$. We assume that $K'_p = U(2)(\mathbb{Z}_p)$ for all $p \notin S_f$, and likewise

for the other groups. The local components of these groups for $w|p \in S_f$ will be specified in the proof of Proposition 5.2. We define congruence subgroups $K'(n)$, etc. of these groups for n relatively prime to S_f in the usual way, and recall that n will only be divisible by primes that split in E .

We let \tilde{P} be the standard parabolic subgroup of $GL(4, E)$ with Levi $\tilde{L} = GL(2, E) \times GL(2, E)$, and let P be the corresponding standard parabolic subgroup of $U(4)$. We let P' be the standard Borel subgroup of $U(2)$.

5.1. Controlling a single parameter. We fix an even parameter $\phi^N \in \tilde{\Phi}_{\text{sim}}(2)$, and let $\psi \in \Phi(U(4), \xi_+)$ be the unique parameter with $\psi^N = \nu(2) \boxtimes \phi^N$. We let ϕ^N correspond to a conjugate self-dual cuspidal automorphic representation μ of $GL(2, \mathbb{A}_E)$. We assume that μ is tempered at all places; as before, this is done only for simplicity. We let $\psi' \in \Psi(U(2), \xi_-)$ be the unique parameter with $\psi'^N = \phi^N$. We shall prove the following bound for the contribution of Π_ψ to (8).

Proposition 5.2. *There is a choice of K'_p for $p \in S_f$, depending only on K , such that*

$$(9) \quad \sum_{\pi \in \Pi_\psi} \dim \pi_f^{K(n)} \ll [K' : (K' \cap P'(\mathbb{A}_f))K'(n)][K : (K \cap P(\mathbb{A}_f))K(n)] \sum_{\pi' \in \Pi_{\psi'}} \dim \pi_f'^{K'(n)}.$$

We begin the proof of Proposition 5.2 with Lemmas 5.3–5.6 below, which control the LHS of (9) in terms of μ .

Lemma 5.3. *Let $p \notin S_f$ be split in E , and let $w|p$. Let $\Pi_{\psi_p} = \{\pi_p\}$. We have*

$$\dim \pi_p^{K_p(n)} \leq [K_p : (K_p \cap P_p)K_p(n)](\dim \mu_w^{\tilde{K}'_w(n)})^2.$$

Proof. Under the identification $U(4)_p \simeq GL(4, E_w)$, π_p is the Langlands quotient of the representation ρ_{ψ_w} of $GL_4(E_w)$ induced from the representation $\mu_w(x_1)|\det(x_1)|^{1/2} \otimes \mu_w(x_2)|\det(x_2)|^{-1/2}$ of \tilde{P}_w . We have

$$\dim \pi_p^{K_p(n)} \leq \dim \rho_{\psi_w}^{\tilde{K}_w(n)}.$$

The restriction of ρ_{ψ_w} to \tilde{K}_w is isomorphic to the induction of $\mu_w(x_1) \times \mu_w(x_2)$ from $\tilde{K}_w \cap \tilde{P}_w$ to \tilde{K}_w . We see that

$$\begin{aligned} \dim \rho_{\psi_w}^{\tilde{K}_w(n)} &= [\tilde{K}_w : (\tilde{K}_w \cap \tilde{P}_w)\tilde{K}_w(n)] \dim(\mu_w \times \mu_w)^{\tilde{L}_w \cap \tilde{K}_w(n)} \\ &= [\tilde{K}_w : (\tilde{K}_w \cap \tilde{P}_w)\tilde{K}_w(n)](\dim \mu_w^{\tilde{K}'_w(n)})^2, \end{aligned}$$

which is equivalent to the lemma. □

Lemma 5.4. *Let $p \notin S_f$ be nonsplit in E , and let $w|p$. We have*

$$\sum_{\pi_p \in \Pi_{\psi_p}} \dim \pi_p^{K_p} \leq \dim \mu_w^{\tilde{K}'_w}.$$

Proof. Identify \tilde{K}_w with a subgroup of $G(4)_p$. The twisted fundamental lemma implies that the functions 1_{K_p} and $1_{\tilde{K}_w \rtimes \theta}$ are related by transfer, so we may apply the character identity of [9], Theorem 3.2.1 (b) with $s = e$ to obtain

$$\sum_{\pi_p \in \Pi_{\psi_p}} \dim \pi_p^{K_p} = \sum_{\pi_p \in \Pi_{\psi_p}} \text{tr}(\pi_p(1_{K_p})) = \text{tr}(\tilde{\pi}_{\psi_p}(1_{\tilde{K}_w \rtimes \theta})).$$

The twisted trace $\text{tr}(\tilde{\pi}_{\psi_p}(1_{\tilde{K}_w \rtimes \theta}))$ is equal to the trace of $\tilde{\pi}_{\psi_p}(\theta)$ on $\pi_{\psi_p}^{\tilde{K}_w}$, so we have

$$\text{tr}(\tilde{\pi}_{\psi_p}(1_{\tilde{K}_w \rtimes \theta})) \leq \dim \pi_{\psi_p}^{\tilde{K}_w}.$$

Under the identification $G(4)_p \simeq GL(4, E_w)$, π_{ψ_p} is the Langlands quotient of the representation ρ_{ψ_w} induced from $\mu_w(x_1)|\det(x_1)|^{1/2} \otimes \mu_w(x_2)|\det(x_2)|^{-1/2}$. We therefore have

$$\dim \pi_{\psi_p}^{\tilde{K}_w} \leq \dim \rho_{\psi_w}^{\tilde{K}_w} \leq \dim \mu_w^{\tilde{K}'_w},$$

and the result follows. □

Lemma 5.5. *Let $p \in S_f$, and let $w|p$. There is a choice of \tilde{K}'_w , depending only on K , such that*

$$\sum_{\pi_p \in \Pi_{\psi_p}} \dim \pi_p^{K_p} \ll \dim \mu_w^{\tilde{K}'_w}.$$

Proof. Suppose that p is nonsplit. Let $\tilde{1}_{K_p} \in \tilde{\mathcal{H}}_p(4)$ be a function corresponding to 1_{K_p} under twisted transfer. Reasoning as in Lemma 5.4 gives

$$\sum_{\pi_p \in \Pi_{\psi_p}} \dim \pi_p^{K_p} = dg_p(K_p)^{-1} \text{tr}(\tilde{\pi}_{\psi_p}(\tilde{1}_{K_p})).$$

If we choose $\tilde{K}_w \subset GL(4, E_w) \simeq G(4)_p$ to be a compact open subgroup such that $\tilde{1}_{K_p}$ is bi-invariant under \tilde{K}_w , we have

$$\text{tr}(\tilde{\pi}_{\psi_p}(\tilde{1}_{K_p})) \ll \dim \pi_{\psi_p}^{\tilde{K}_w}.$$

Under the identification $G(4)_p \simeq GL(4, E_w)$, π_{ψ_p} is the Langlands quotient of the representation ρ_{ψ_w} induced from $\mu_w(x_1)|\det(x_1)|^{1/2} \otimes \mu_w(x_2)|\det(x_2)|^{-1/2}$. Choose \tilde{K}'_w so that the product $\tilde{K}'_w \times \tilde{K}'_w$ is contained in \tilde{K}_w . We then have

$$\dim \pi_{\psi_p}^{\tilde{K}_w} \leq \dim \rho_{\psi_w}^{\tilde{K}_w} \ll (\dim \mu_w^{\tilde{K}'_w})^2.$$

Bounding $\dim \mu_w^{\tilde{K}'_w}$ by a constant depending on \tilde{K}'_w , and hence K_p , completes the proof for p nonsplit. The proof in the split case follows in exactly the same way using the explicit description of π_p . □

Lemma 5.6. *If $p \notin S_f$ is split and $w|p$, we have*

$$\dim \mu_w^{\tilde{K}'_w(n)} \leq [\tilde{K}'_w : (\tilde{K}'_w \cap \tilde{P}'_w) \tilde{K}'_w(n)] = [K'_p : (K'_p \cap P'_p) K'_p(n)].$$

Proof. If μ_w is a principal series representation or a twist of Steinberg, this is immediate. If μ_w is supercuspidal, this follows by examining the construction of supercuspidal representations given in §7.A. of [6]. □

Let $S_{E/\mathbb{Q}}$ be a set of finite places of E that contains exactly one place above every finite place of \mathbb{Q} . Combining Lemmas 5.3–5.6 gives

$$\sum_{\pi \in \Pi_\psi} \dim \pi_f^{K(n)} \ll [K' : (K' \cap P'(\mathbb{A}_f))K'(n)][K : (K \cap P(\mathbb{A}_f))K(n)] \prod_{w \in S_{E/\mathbb{Q}}} \dim \mu_w^{\tilde{K}'_w(n)}.$$

Proposition 5.2 now follows from the lemma below.

Lemma 5.7. *There is a choice of K'_p for $p \in S_f$, depending only on K , such that*

$$\prod_{w \in S_{E/\mathbb{Q}}} \dim \mu_w^{\tilde{K}'_w(n)} \ll \sum_{\pi' \in \Pi_{\psi'}} \dim \pi'_f{}^{K'(n)}.$$

Proof. We may factorise the RHS as

$$\sum_{\pi' \in \Pi_{\psi'}} \dim \pi'_f{}^{K'(n)} = \prod_p \sum_{\pi'_p \in \Pi_{\psi'_p}} \dim \pi'_p{}^{K'_p(n)}.$$

Let p be an arbitrary prime, and $w|p$. It suffices to show that

$$(10) \quad \dim \mu_w^{\tilde{K}'_w(n)} \leq \sum_{\pi'_p \in \Pi_{\psi'_p}} \dim \pi'_p{}^{K'_p(n)}$$

if $p \notin S_f$, and that if $p \in S_f$ the same inequality holds with a constant factor depending only on \tilde{K}' , and hence K .

If p is split, then $\Pi_{\psi'_p}$ contains a single representation that is isomorphic to $\mu_w \otimes \chi_w^{-1}$ under the identification $U(2)_p \simeq GL(2, E_w)$, and (10) is immediate.

Suppose that $p \notin S_f$ is nonsplit. The definition of ψ'_p implies that if $\xi_- : {}^L U(2) \rightarrow {}^L G(2)$, the representation of $G(2)_p \simeq GL(2, E_w)$ associated to $\xi_- \circ \psi'_p \in \Psi_p(2)$ is μ_w . We let $\tilde{\mu}_w$ denote the canonical extension of μ_w to a representation of $\tilde{G}^+(2)$, and identify \tilde{K}'_w with a subgroup of $G(2)_p$. Theorem 3.2.1 of [9] and the twisted fundamental lemma give

$$(11) \quad \text{tr}(\tilde{\mu}_w(1_{\tilde{K}'_w \rtimes \theta})) = \sum_{\pi'_p \in \Pi_{\psi'_p}} \text{tr}(\pi'_p(1_{K'_p})) = \sum_{\pi'_p \in \Pi_{\psi'_p}} \dim \pi'_p{}^{K'_p}.$$

The LHS of (11) is equal to the trace of $\tilde{\mu}_w(\theta)$ on $\mu_w^{\tilde{K}'_w}$. If $\dim \mu_w^{\tilde{K}'_w} = 0$ then both sides of (11) are 0, and (10) holds. If $\dim \mu_w^{\tilde{K}'_w} = 1$, then $\theta^2 = 1$ implies that $\text{tr}(\tilde{\mu}_w(1_{\tilde{K}'_w \rtimes \theta})) = \pm 1$. Positivity implies that we must take the plus sign so that (10) also holds.

Suppose that $p \in S_f$ is nonsplit, and suppose that the LHS of (10) is nonzero. Up to twist, there are only finitely many possibilities for μ_w that are supercuspidal or Steinberg, and we may deal with these cases by simply choosing K'_p so that (10) is true in each case. If

μ_w is induced from a unitary character of the Borel, then $\Pi_{\psi'_p}$ is described explicitly in §11.4 of [10] and (10) follows easily from this description. \square

5.2. Summing over parameters. We define $\Phi_{\text{rel}} \subset \tilde{\Phi}_{\text{sim}}(2)$ to be the set of even parameters ϕ^N such that

$$\phi_{\infty}^N : z \mapsto \begin{pmatrix} z/\bar{z} & \\ & \bar{z}/z \end{pmatrix}.$$

It may be shown in the same way as Lemma 4.6 that if $\psi \in \Psi(U(4), \xi_+)$ satisfies $\psi^N = \nu(2) \boxtimes \phi^N$ with $\phi^N \in \tilde{\Phi}_{\text{sim}}(2)$, and $\pi \in \Pi_{\psi_{\infty}}$ satisfies $H^*(\mathfrak{g}, K; \pi) \neq 0$, then $\phi^N \in \Phi_{\text{rel}}$. If $\phi^N \in \Phi_{\text{rel}}$ corresponds to the cusp form μ , then $\mu_{\infty} \times \chi_{\infty}$ has infinitesimal character $(3/2 + t, -1/2 + t; -3/2 - t, 1/2 - t)$, and Theorem 1.2 of [5] implies that μ is tempered at all places. It follows from this discussion that

$$(12) \quad \sum_{\substack{\psi \in \Psi(U(4), \xi_+) \\ \psi^N = \nu(2) \boxtimes \phi^N}} \sum_{\pi \in \Pi_{\psi}} h^3(\mathfrak{g}, K; \pi_{\infty}) \dim \pi_f^{K(n)} \ll \sum_{\substack{\psi^N = \nu(2) \boxtimes \phi^N \\ \phi^N \in \Phi_{\text{rel}}}} \sum_{\pi \in \Pi_{\psi}} \dim \pi_f^{K(n)}.$$

Applying Proposition 5.2 to the sum on the RHS gives

$$(13) \quad \sum_{\substack{\psi^N = \nu(2) \boxtimes \phi^N \\ \phi^N \in \Phi_{\text{rel}}}} \sum_{\pi \in \Pi_{\psi}} \dim \pi_f^{K(n)} \ll [K' : (K' \cap P'(\mathbb{A}_f))K'(n)][K : (K \cap P(\mathbb{A}_f))K(n)] \\ \sum_{\substack{\psi' \in \Psi(U(2), \xi_-) \\ \psi'^N \in \Phi_{\text{rel}}}} \sum_{\pi' \in \Pi_{\psi'}} \dim \pi_f'^{K'(n)}.$$

The restriction on the infinitesimal characters of parameters in Φ_{rel} implies that there is a finite set of representations Ξ_{∞} of $U(2)_{\infty}$ such that if $\psi'^N \in \Phi_{\text{rel}}$, then all the representations in $\Pi_{\psi'_{\infty}}$ are in Ξ_{∞} . Because Φ_{rel} consists of simple generic parameters, every $\pi' \in \Pi_{\psi'}$ occurs in $L^2_{\text{disc}}(U(2)(\mathbb{Q}) \backslash U(2)(\mathbb{A}))$ with multiplicity one. If we define $X(n) = U(2)(\mathbb{Q}) \backslash U(2)(\mathbb{A}) / K'(n)$, this gives

$$(14) \quad \sum_{\substack{\psi' \in \Psi(U(2), \xi_-) \\ \psi'^N \in \Phi_{\text{rel}}}} \sum_{\pi' \in \Pi_{\psi'}} \dim \pi_f'^{K'(n)} \leq \sum_{\substack{\pi' \in L^2_{\text{disc}}(U(2)(\mathbb{Q}) \backslash U(2)(\mathbb{A})) \\ \pi'_{\infty} \in \Xi_{\infty}}} \dim \pi_f'^{K'(n)} \\ = \sum_{\pi_{\infty} \in \Xi_{\infty}} m(\pi_{\infty}, X(n)) \\ \ll [K' : K'(n)].$$

Combining (12)–(14) gives

$$\sum_{\substack{\psi \in \Psi(U(4), \xi_+) \\ \psi^N = \nu(2) \boxtimes \phi^N}} \sum_{\pi \in \Pi_\psi} h^3(\mathfrak{g}, K; \pi_\infty) \dim \pi_f^{K(n)}$$

$$\ll [K' : (K' \cap P'(\mathbb{A}_f))K'(n)][K : (K \cap P(\mathbb{A}_f))K(n)][K' : K'(n)],$$

and applying the formula for the order of $GL(N)$ over a finite field completes the proof.

5.3. The case of composite ϕ^N . We now suppose that $\phi^N = \phi_1^N \boxplus \phi_2^N$, where $\phi_i^N \in \tilde{\Phi}(1)$ correspond to conjugate self-dual characters μ_i . We may prove the following analogue of Proposition 5.2.

Proposition 5.8. *There is a choice of $\tilde{K}_{1,w}$ for $w|p \in S_f$, depending only on K , such that*

$$\sum_{\pi \in \Pi_\psi} \dim \pi_f^{K(n)} \ll [K : (K \cap P(\mathbb{A}_f))K(n)] \dim \mu_1^{\tilde{K}_1(n)} \dim \mu_2^{\tilde{K}_1(n)}.$$

Unlike Proposition 5.2, this bound is sharp. The reason for this is that the representation π_{ψ_p} for split p is equivalent to the induction of $\mu_{1,w} \circ \det(x_1)|\det(x_1)|^{1/2} \otimes \mu_{2,w} \circ \det(x_2)|\det(x_2)|^{-1/2}$ from \tilde{P}_w to $GL(4, E_w)$, and it is easy to give a sharp bound for the dimension of invariants under $\tilde{K}_w(n)$, unlike the Speh representations considered before. We obtain a bound of $n^{6+\epsilon}$ for the contribution of these parameters to $h_{(2)}^3(Y(n))$.

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